Final Exam Practice Problems

1. For the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 \\ 7 & 0 \\ 8 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$, calculate:

(i) $6A + 2B^t$.
(ii) $\det(2B \cdot 3A)$.
(iii) ABC^{-1}.

2. Write the solution set to the system of linear equations having the following augmented matrices in reduced row echelon form:

$$A = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right].$$

3. Suppose that U is the 3×3 coefficient matrix of the system of equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ y + 4z &= -1 \\ 5x + 6y &= -5. \end{aligned}$$

(i) Calculate the determinant of U in two ways: By expanding along the third row and expanding along the second column.
(ii) Convert the system of equations into a single matrix equation.
(iii) Solve the matrix equation by finding U^{-1}.
(iv) Write the solution to the original systems of equation.

4. Determine whether or not the following matrices are diagonalizable. If not, explain why. If so, in each case find the diagonalizing matrix P. In these latter cases, check your answer, i.e., if P diagonalizes say A, then verify that $P^{-1}AP$ is a diagonal matrix.

$$A = \begin{bmatrix} -4 & 2 \\ 0 & 6 \end{bmatrix}, B = \begin{bmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -13 & 0 & 16 \\ 0 & 0 & 3 & 0 \\ 0 & -12 & 0 & 15 \end{bmatrix}.$$

In finding the eigenvalues of each matrix, it might be useful to use the following version of the Rational Root Test: If $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ is a polynomial with integer coefficients, then the only possible roots in the rational numbers are $\pm d$ where d divides a_n.

5. For the matrices in problem 4 that are diagonalizable, say, for example, A, write a formula for A^n and e^A.

6. For the system of first order linear differential equations:

$$\begin{aligned} x_1'(t) &= 2x_1(t) \\ x_2'(t) &= x_1(t) + x_2(t) \\ x_3'(t) &= -x_1(t) + x_3(t) \end{aligned}$$

with initial conditions $x_1(0) = -1, x_2(0) = -2, x_3(0) = 2$.

(i) Convert the system of equations into a single matrix differential equation with initial condition, clearly indicating the terms the matrix equation.
(ii) Solve the equation in (ii) using the exponential function.
(iii) Write the solutions to the original system.

7. Given the linear recurrence relation $a_{k+2} = a_{k+1} + 2a_k$ and the initial conditions $a_0 = 2, a_1 = 7$,

(i) Write out the first five terms of the sequence.

(ii) Find a formula for a_k, for all k, that is not recursive.

8. Let U be the subspace of $\mathbb{R}^4$ spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 6 \\ -2 \\ 8 \\ 6 \end{bmatrix}$, $v_4 = \begin{bmatrix} 8 \\ -1 \\ 9 \\ 8 \end{bmatrix}$.

   (i) Find a basis for U.

   (ii) Determine if the vectors $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}$ belong to U.

9. Find an orthonormal basis for each of the eigenspaces of the diagonalizable matrices in problem 4.

10. Find the best approximation to a solution of the system

$$\begin{aligned} 6x + 4y &= 14 \\ 8x - 2y &= 4 \\ -2x + 4y &= 4. \end{aligned}$$

11. Given the data points (2,6), (-1,4), (-2,3), find:
   (i) The line best fitting the data.
   (ii) The quadratic polynomial best fitting the data.